Inequality with altitudes, circumradius and inradius in a triangle.
https://www.linkedin.com/feed/update/urn:li:activity:6640113087653515264
In any triangle with altitudes h_a, h_b, h_c, whereR, r are its circumradius and inradius respectively, show that

$$(h_a - r)(h_b - r)(h_c - r) \leq 4Rr^2$$

Solution by Arkady Alt, San Jose ,California, USA.

Let s be semiperimeter of the triangle. Since $h_a = \dfrac{2rs}{a}$ then $h_a - r = \dfrac{2rs}{a} - r =$ $\dfrac{r(2s-a)}{a} = \dfrac{r(b+c)}{a}$ and, therefore, $\prod(h_a - r) \leq 4Rr^2 \Leftrightarrow \prod \dfrac{r(b+c)}{a} \leq 4Rr^2 \Leftrightarrow$ $\dfrac{r^3(a+b)(b+c)(c+a)}{abc} \leq 4Rr^2 \Leftrightarrow \dfrac{r(a+b)(b+c)(c+a)}{abc} \leq 4R \Leftrightarrow$ $\dfrac{r(a+b)(b+c)(c+a)}{4Rrs} \leq 4R \Leftrightarrow (a+b)(b+c)(c+a) \leq 16R^2 s \Leftrightarrow$ $(a+b+c)(ab+bc+ca) - abc \leq 16Rs \Leftrightarrow 2s(ab+bc+ca) - 4Rrs \leq 16R^2 s \Leftrightarrow$ $ab+bc+ca \leq 8R^2 + 2Rr \Leftrightarrow s^2 + 4Rr + r^2 \leq 8R^2 + 2Rr \Leftrightarrow s^2 \leq 8R^2 - 2Rr - r^2$.

Since $s^2 \leq 4R^2 + 4Rr + 3r^2$ (Gerretsens's Inequality) and $2r \leq R$ (Euler's Inequality) we obtain $8R^2 - 2Rr - r^2 - s^2 \geq 8R^2 - 2Rr - r^2 - (4R^2 + 4Rr + 3r^2) =$ $4R^2 - 6Rr - 4r^2 = (R - 2r)(2R + r) \geq 0$.

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